FINITE PLANE DEFORMATIONS OF AN INCOMPRESSIBLE MATERIAL

(KONECHNYE PLOSKIE DEFORMATSII NESZHIMAEMOGO MATERIALA)

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In the analysis of large strains and unlimited displacements, the compatibility conditions for translational displacements and rotations of elements of a deforming body will be formulated in terms of invariant deformation characteristics. The equilibrium equations are so transformed that it becomes possible to introduce a stress function in terms of whose derivatives the invariant characteristics of the true stress can be very simply expressed. With the aid of complex coordinates the mathematical formulation of the problem can be presented in a compact form. The solutions of actual problems can be found by successive approximations where every step reduces to the solution of the classical biharmonic problem. In conclusion, as an example, the stress concentrations near a circular cylindrical cavity are studied.

1. Geometry of the Plane Deformation of a Continuous Body. Two states of the continuous body, the original and the deformed, are studied. The position of a material particle in the original state is given by the Cartesian coordinates x_1, x_2, x_3. The displacement vector $\mathbf{u} = u_k\mathbf{i}_k$ denotes the passage of the body from the original to the deformed state to be analyzed. In this paper repeated indices indicate summation on these indices from one to three. In plane strain, only the displacement components u_1 and u_2 are different from zero, and they are assumed to be functions of x_1 and x_2 only.

The linear components of the strain tensor and the components of the vector ω are computed from formulas [1]

$$\begin{aligned} &\varepsilon_{11} = u_{1,1}, \ \varepsilon_{22} = u_{2,2}, \qquad 2\varepsilon_{12} = u_{1,2} + u_{2,1} \\ &\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0, \qquad \omega_1 = \omega_2 = 0, \qquad 2\omega_3 = u_{2,1} - u_{1,2} \end{aligned} \tag{1.1}$$

The indices 1 and 2 after a comma denote partial differentiation with respect to the variables x_1 and x_2, for instance

$$\frac{\partial u_1}{\partial x_2} = u_{1,2}, \qquad \frac{\partial u_2}{\partial x_2} = u_{2,2}$$

The projections α_{ks} of the vectors $\mathbf{a}_k$ which are orthogonal to the material coordinate surface elements in the deformed state are expressed in terms of the derivatives of the displacements by the formulas

$$\begin{aligned}\alpha_{11} &= 1 + \varepsilon_{22}, \quad \alpha_{12} = -(\varepsilon_{12} - \omega_3), \quad \alpha_{13} = \alpha_{23} = \alpha_{31} = \alpha_{32} = 0 \\ \alpha_{21} &= -(\varepsilon_{12} + \omega_3), \quad \alpha_{22} = 1 + \varepsilon_{11}, \quad \alpha_{33} = (1 + \varepsilon_{11})(1 + \varepsilon_{22}) - \varepsilon_{12}^2 + \omega_3^2\end{aligned} \tag{1.2}$$

Finally, we shall introduce expressions, in terms of derivatives of the displacements, for the coefficients of distortion s_k of the areas of the coordinate surface elements, elongations λ_{ss} of the coordinate fibers, and the magnitudes Δ of the relative change of volume:

$$\begin{aligned}&\lambda_{11}^2 = s_2^2 = (1 + \varepsilon_{11})^2 + (\varepsilon_{12} + \omega_3)^2 \\ &\lambda_{22}^2 = s_1^2 = (1 + \varepsilon_{22})^2 + (\varepsilon_{12} - \omega_3)^2 \qquad s_3^2 = \alpha_{33}^2 = (1 + \Delta)^2\end{aligned} \tag{1.3}$$

The relations introduced here show that the geometry of the deformed state of the neighborhood of an arbitrary point of the body and the orientation of this neighborhood is completely determined by the four derivatives $u_{k,s}$. One can also choose another set of four parameters, where an important part is played by the strain invariants, as a set of coordinates of the deformed state of the neighborhood of a point.

Denote the angle between the first principal direction of the strain in the original state and the first coordinate direction by θ. The elongations of the principal fibers will be denoted by λ_s, where $\lambda_3 = 1$. Assume that during the deformation the neighborhood of the point rotates as a rigid body around $\mathbf{i}_3$ through an angle ω.

On the basis of the geometric interpretation of the parameters θ, ω, λ_1, and λ_2, it may be stated that these parameters are sufficient for the determination of the deformed state and the orientation of the neighborhood of a particle in the deformed state. In particular, ϵ_{ik} and ω_3 can be expressed in terms of the parameters θ, ω, λ_1 and λ_2. With this goal in mind let us study the unit vectors $\mathbf{I}_k$, which determine the directions of the principal fibers in the original state

$$\mathbf{I}_1 = \cos\theta \mathbf{i}_1 + \sin\theta \mathbf{i}_2, \qquad \mathbf{I}_2 = -\sin\theta \mathbf{i}_1 + \cos\theta \mathbf{i}_2, \qquad \mathbf{I}_3 = \mathbf{i}_3$$

and the unit vectors $\mathbf{I}_k'$, which determine the directions of the same material fibers in the deformed state

$$\begin{aligned}\mathbf{I}_1' &= \cos(\theta + \omega)\,\mathbf{i}_1 + \sin(\theta + \omega)\,\mathbf{i}_2 \\ \mathbf{I}_2' &= -\sin(\theta + \omega)\,\mathbf{i}_1 + \cos(\theta + \omega)\,\mathbf{i}_2, \qquad \mathbf{I}_3' = \mathbf{i}_3\end{aligned} \tag{1.4}$$

On the other hand, the vectors (1.4) can also be determined from the fundamental relation of the general deformation theory [2]:

$$I_n' = \frac{1}{\lambda_n}[I_n + \omega_3 i_3 \times I_n + ((\varepsilon_{ik}))I_n] \qquad (n = 1, 2) \tag{1.5}$$

Here $\mathbf{i}_3 \times \mathbf{I}_n$ denotes the vector cross product, and the symbol $((\epsilon_{ik}))\mathbf{I}_n$ represents the result of a linear transformation of the vector $\mathbf{I}_n$ by means of the tensor $((\epsilon_{ik}))$.

By setting the different expressions of the same vectors equal to each other, the required relations between the variant and the invariant strain characteristics are easily obtained. Let us reduce these relations to a form which is most suitable for further application, and let us solve for the respective derivatives:

$$\begin{aligned} 2(1+\varepsilon_{11}) &= (\lambda_1+\lambda_2)\cos\omega + (\lambda_1-\lambda_2)\cos(2\theta+\omega) = 2(1+u_{1,1}) \\ 2(\varepsilon_{12}-\omega_3) &= -(\lambda_1+\lambda_2)\sin\omega + (\lambda_1-\lambda_2)\sin(2\theta+\omega) = 2u_{1,2} \\ 2(1+\varepsilon_{22}) &= (\lambda_1+\lambda_2)\cos\omega - (\lambda_1-\lambda_2)\cos(2\theta+\omega) = 2(1+u_{2,2}) \\ 2(\varepsilon_{12}+\omega_3) &= (\lambda_1+\lambda_2)\sin\omega + (\lambda_1-\lambda_2)\sin(2\theta+\omega) = 2u_{2,1} \end{aligned} \tag{1.6}$$

From this, incidentally, the geometrical meaning of the vector ω in plane strain can easily be established:

$$2\omega_3 = (\lambda_1+\lambda_2)\sin\omega$$

If the simplifications inherent in the formation of the linear deformation theory were carried out, then (1.6) would reduce to expressions for the strain tensor components with respect to arbitrary axes in terms of the principal values of the strain tensor.

2. Compatibility Conditions for Translational Displacements of Particles of the Body and Conditions of Continuity of Small Rotations. As is well known [3], in the classical theory of the requirements of continuity of translational displacements and small rotations of the elements of the body lead to the strain compatibility conditions: the St Venant identities and the Beltrami equations. The St Venant identities retain their validity as conditions for the continuity of the displacements and also the functions ω_k for deformations which are not small. In order to simplify later applications of the compatibility conditions for plane strain in problems formulated in terms of stress, the strain compatibility conditions will be written in terms of invariant strain characteristics, which preserve the geometrical meaning with unlimited strains and displacements. It will here be assumed that the constancy of the volume during deformation is a physical property of the deforming body:

$$\Delta = 0 \tag{2.1}$$

The logarithmic elongations of change of shape [deviators] are expressed in terms of the intensity ϑ_i and the phase β of the change of shape [strain deviator] by the formulas

$$\ln \frac{\lambda_k}{(1+\Delta)^{1/3}} = \sqrt{2}\, \vartheta_i \cos\beta_k, \quad \beta_1 = \beta, \quad \beta_2 = \beta + \tfrac{2}{3}\pi, \quad \beta_3 = \beta - \tfrac{2}{3}\pi \tag{2.2}$$

Therefore, the phase of the plane strain of the incompressible material is known to be determined:

$$\beta = \tfrac{1}{6}\pi \tag{2.3}$$

Thus the principal elongations of the incompressible material in plane strain are determined by only one parameter, the intensity of change of shape [strain deviator] :

$$\lambda_1 + \lambda_2 = 2\cosh\sqrt{1.5}\,\vartheta_i, \quad \lambda_1 - \lambda_3 = 2\sinh\sqrt{1.5}\,\vartheta_i, \quad \lambda_3 = 1 \tag{2.4}$$

In the case of plane strain of the incompressible material, the deformation parameters ϵ_{ik} and ω_s can be expressed by using formulas (1.6) and (2.4), in terms of the invariant strain characteristics ϑ_i and ω and the variant quantity θ. Comparing the mixed second derivatives of u_1 and u_2, computed according to (1.6), we obtain the conditions of continuity of translational displacements of the body elements:

$$2(\theta+\omega)_{,1} = 2\cosh\vartheta\,\theta_{,1} - (\cosh\vartheta)_{,2} + (\sin 2\theta \sinh\vartheta)_{,1} - (\cos 2\theta \sinh\vartheta)_{,2} \tag{2.5}$$

$$2(\theta+\omega)_{,2} = 2\cosh\vartheta\,\theta_{,2} + (\cosh\vartheta)_{,1} - (\sin 2\theta \sinh\vartheta)_{,2} - (\cos 2\theta \sinh\vartheta)_{,1}$$

where

$$\vartheta = 2\sqrt{1.5}\,\vartheta_i = 2\ln(1+\eta), \qquad \eta = \lambda_1 - 1 \tag{2.6}$$

The rotations of the body elements can be determined from equations (2.5) if it is assumed that the pure strain characteristics θ and are known. The conditions for the continuity of these rotations can easily be found from (2.5):

$$\begin{aligned}&(\cos 2\theta \sinh\vartheta)_{,22} - (\cos 2\theta \sinh\vartheta)_{,11} - 2(\sin 2\theta \sinh\vartheta)_{,12} + \\ &+ (\cosh\vartheta)_{,11} + (\cosh\vartheta)_{,22} + 2\theta_{,2}(\cosh\vartheta)_{,1} - 2\theta_{,1}(\cosh\vartheta)_{,2} = 0\end{aligned} \tag{2.7}$$

Relation (2.7) does not differ in contents from the said St Venant identity. If a system of simplifications corresponding to the theory of small deformations is adopted, i.e. if $\sinh\vartheta = \vartheta$, $\cosh\vartheta = 1$, then an equation is obtained which also agrees in form with the St Venant equation.

3. Stress-Strain Relations, Equilibrium Equations. The differential equations of equilibrium in Lagrange coordinates [2] contain as the fundamental unknown functions the generalized stresses Σ_{mn}, which are expressed in terms of the coefficients of distortion of the coordinate surface elements s_m and the contravariant components σ_{mn} of the true stress tensor, referred to the deformed system of coordinate fibers. If a normal to some material surface element coincided with $\mathbf{i}_n$ before deformation, then the normal to this same material surface element will be $\mathbf{b}_n$ after deformation, hence

$$s_k \mathbf{b}_k = \alpha_{k2}\mathbf{i}_1 + \alpha_{k2}\mathbf{i}_2 \qquad (k = 1,2) \tag{3.1}$$

Let us recall that σ_{mn} represents the orthogonal projection of the stress [traction] vector, acting on a surface element whose normal is $\mathbf{b}_m$ in the direction $\mathbf{b}_n$. It is useful to compute in advance the projections of the unit vectors of the normals to the coordinate surface elements in the deformed state onto the directions of the principal fibers in the deformed state. Let

$$\mathbf{b}_k = b_{k1}\mathbf{I}_1' + b_{k2}\mathbf{I}_2' \tag{3.2}$$

then on the basis of (3.1) and (1.4) we find

$$\begin{aligned} b_{11}s_1 &= a_{11} = \alpha_{11}\cos(\theta+\omega) + \alpha_{12}\sin(\theta+\omega) \\ b_{12}s_1 &= a_{12} = -\alpha_{11}\sin(\theta+\omega) + \alpha_{12}\cos(\theta+\omega) \\ b_{21}s_2 &= a_{21} = \alpha_{21}\cos(\theta+\omega) + \alpha_{22}\sin(\theta+\omega) \\ b_{22}s_2 &= a_{22} = -\alpha_{21}\sin(\theta+\omega) + \alpha_{22}\cos(\theta+\omega) \end{aligned} \tag{3.3}$$

We will assume henceforward that the principal directions of the stresses in the state under consideration coincide with the directions of the principal fibers in the deformed state. Then the stress [traction] vectors σ_m, acting on the surface elements whose normals are $\mathbf{b}_m$, can easily be expressed in terms of the true principal stresses σ_1 and σ_2. Thus we have

$$\sigma_{mn} = \sigma_m \cdot \mathbf{b}_n$$

and hence the use of (1.2) leads to the following relations:

$$\Sigma_{11} = \sigma_1 a_{11}^2 + \sigma_2 a_{12}^2, \qquad \Sigma_{22} = \sigma_1 a_{21}^2 + \sigma_2 a_{22}^2, \qquad \Sigma_{12} = \sigma_1 a_{11}a_{21} + \sigma_2 a_{12}a_{22} \tag{3.4}$$

With formulas (3.3) available, the generalized stresses can be computed in terms of the principal stresses and the symmetric characteristics of the deformed state. The principal stresses can in turn be expressed in terms of symmetric stress invariants: the hydrostatic stress σ, the octahedral shearing stress τ_i, and the stress phase ϕ - by means of the formulas

$$\sigma_k = \sigma + \sqrt{2}\,\tau_i \cos\varphi_k, \qquad \varphi_1 = \varphi, \qquad \varphi_2 = \varphi + \tfrac{2}{3}\pi, \qquad \varphi_3 = \varphi - \tfrac{2}{3}\pi$$

Since the third principal stress - the axial stress σ_3 - is determined from the value of the hydrostatic stress only, the phase of the true stresses in the plane deformed state is known to be $\phi = 1/6\ \pi$.

This the examination of the physical properties of a body undergoing plane deformations reduces to the experimental determination of the dependence of the octahedral shearing stress on the intensity of the change of shape [strain deviator] for different values of the mean normal stress. Since only a limited range of variation of the hydrostatic stresses will be studied, it will be assumed henceforward that the following relation is known from experiments:

$$\sigma_i = \tau_i(\vartheta_i) \tag{3.5}$$

It is to hold good universally for the range of hydrostatic stresses studied. The intensity of the change of shape [strain deviator] will in turn be assumed to depend only on the octahedral shearing stress.

Assuming that the phase of the true stresses can be determined, the expressions for the principal true stresses are first obtained.

$$\sigma_1 = \sigma + \sqrt{1.5}\,\tau_i, \qquad \sigma_2 = \sigma - \sqrt{1.5}\,\tau_i, \qquad \sigma_3 = \sigma \tag{3.6}$$

and then the expressions for the generalized stresses are also determined:

$$\begin{aligned} \Sigma_{11} + \Sigma_{22} &= 2\sigma \cosh \vartheta - 2\sqrt{1.5}\,\tau_i \sinh \vartheta \\ \Sigma_{11} - \Sigma_{22} &= 2\cos 2\theta\,(-\sigma \sinh \vartheta + \sqrt{1.5}\,\tau_i \cosh \vartheta) \\ \Sigma_{12} &= \sin 2\theta\,(-\sigma \sinh \vartheta + \sqrt{1.5}\,\tau_i \cosh \vartheta) \end{aligned} \tag{3.7}$$

The general equilibrium equations in Lagrange coordinates [2], as applied to the case of plane strain in an incompressible material, are considerably simplified and presented as follows:

$$\begin{aligned} \Sigma_{11,1} + \Sigma_{12,2} + \Sigma_{11} f_{11} + \Sigma_{22} f_{12} + \Sigma_{12}(f_{13} + f_{14}) &= 0 \\ \Sigma_{12,1} + \Sigma_{22,2} + \Sigma_{11} f_{12} + \Sigma_{22} f_{22} + \Sigma_{12}(f_{23} + f_{24}) &= 0 \end{aligned}$$

if it is assumed that inertia terms are absent.

Here the following notation was used:

$$\begin{aligned} f_{11} &= \tfrac{1}{2}\cos 2\theta\,\vartheta_{,1} - \sinh \vartheta \sin 2\theta\,(\theta + \omega)_{,1} \\ f_{12} &= \tfrac{1}{2}\sin 2\theta\,\vartheta_{,2} + \theta_{,2} + (-\cosh \vartheta + \sinh \vartheta \cos 2\theta)(\theta + \omega)_{,2} \\ f_{13} &= \tfrac{1}{2}\sin 2\theta\,\vartheta_{,1} + \theta_{,1} + (-\cosh \vartheta + \sinh \vartheta \cos 2\theta)(\theta + \omega)_{,1} \\ f_{14} &= \tfrac{1}{2}\cos 2\theta\,\vartheta_{,2} - \sinh \vartheta \sin 2\theta\,(\theta + \omega)_{,2} \\ f_{21} &= \tfrac{1}{2}\sin 2\theta\,\vartheta_{,1} - \theta_{,1} + (\cosh \vartheta + \sinh \vartheta \cos 2\theta)(\theta + \omega)_{,1} \\ f_{22} &= -\tfrac{1}{2}\cos 2\theta\,\vartheta_{,2} + \sinh \vartheta \sin 2\theta\,(\theta + \omega)_{,2} \end{aligned}$$

$$f_{23}=\frac{1}{2}\sin 2\theta_{,2}\,\vartheta_{,2}-\theta_{,2}+(\cosh\vartheta+\sinh\vartheta\cos 2\theta)(\theta+\omega)_{,2}$$

$$f_{24}=-\frac{1}{2}\cos 2\theta\,\vartheta_{,1}+\sinh\vartheta\sin 2\theta\,(\theta+\omega)_{,1}$$

If it is assumed that the displacement compatibility conditions (2.5) and the expressions for the generalized stresses (3.7) hold good, then the differential equations of equilibrium can be written in the form of a homogeneous algebraic system with two differential operators operating on the stress invariants with a determinant different from zero. The only possible trivial solution of this system represents a new form of the equilibrium equations:

$$\begin{aligned}&\left(\frac{\sigma}{\sqrt{1,5}}-f+\tau_i\cos 2\theta\right)_{,1}+(\tau_i\sin 2\theta)_{,2}=0\\&(\tau_i\sin 2\theta)_{,1}+\left(\frac{\sigma}{\sqrt{1.5}}-f-\tau_i\cos 2\theta\right)_{,2}=0\end{aligned}\tag{3.8}$$

where

$$f=2\sqrt{1.5}\int_0^{\vartheta_i}\tau_i(\vartheta_i)\,d\vartheta_i\tag{3.9}$$

The structure of the equilibrium equations (3.8) is similar to that of the corresponding equations of the classical plane problem, and it thus becomes obvious that it is possible to introduce a stress function U. Satisfying the equilibrium equations, we let

$$\begin{aligned}&\frac{\sigma}{\sqrt{1.5}}-f+\tau_i\cos 2\theta=pU_{,22}\\&\frac{\sigma}{\sqrt{1.5}}-f-\tau_i\cos 2\theta=pU_{,11}\end{aligned}\qquad \tau_i\sin 2\theta=-pU_{,12}\tag{3.10}$$

Here, for convenience, we have introduced a constant multiplier p, which represents the characteristic stress in every problem. Simple algebraic operations lead to a somewhat different form of equations (3.10):

$$\begin{aligned}&\sin 2\theta=-\frac{p}{\tau_i}U_{,12},\quad \cos 2\theta=\frac{1}{2}\frac{p}{\tau_i}(U_{,22}-U_{,11})\\&\tau_i^2=p^2\left[(U_{,12})^2+\frac{1}{4}(U_{,22}-U_{,11})^2\right],\quad \frac{\sigma}{\sqrt{1.5}}=f+\frac{1}{2}p(U_{,22}+U_{,11})\end{aligned}\tag{3.11}$$

Since the deformation intensity is assumed to be uniquely determined in terms of the stress intensity, it can be asserted that not only the stresses but also the deformation characteristics θ and can be determined from the stress function.

For continuity of displacements and rotations corresponding to the system of strains thus determined, the choice of the stress function must be subject to the condition of compatibility of small rotations (2.7). Substitution of expressions (3.5) and (3.11) into compatibility equation (2.7) leads to a differential equation, whose solution is the solution of the problem in terms of the stress function. The actual form of this equation in Cartesian coordinates is not written down here, as below we will introduce a more compact formulation of the problem in complex coordinates.

4. Formulation of the Boundary Conditions. To formulate the geometrical conditions on the boundary of the body we must have at our disposal the physical characteristics of the material (3.5) and the expressions for the displacement components in terms of the stress function. Here we shall be concerned only with the general formulation of the static boundary conditions.

Assume that at every point of the cylindrical surface of the body the true normal stress $p\sigma_n$ and the true shearing stress $p\tau_n$ are given. Here we can note that giving the corresponding conventional stresses $p\lambda_\tau\sigma_n$ and $p\lambda_\tau\tau_n$, where λ_τ is the elongation of the surface fiber, does not introduce any considerable complications into the solution of the problems.

Let α be the angle between the direction of the first coordinate axis and the direction of the outward normal ν to the surface of the body. The direction of the unit vector τ, tangent to the contour of the body, is so chosen that the region occupied by the body is always at the left-hand side when the contour is followed in the positive direction. Thus the vectors of the normal and the tangent are determined by the formulas

$$\nu = n_1\mathbf{i}_1 + n_2\mathbf{i}_2, \qquad \tau = -n_2\mathbf{i}_1 + n_1\mathbf{i}_2, \qquad n_1 = \cos\alpha, \qquad n_2 = \sin\alpha$$

On the basis of the fundamental relation of the deformation theory, we find the direction of the tangent to the contour of the body in the deformed state:

$$\lambda_\tau\tau' = \tau + \omega_2\mathbf{i}_3 \times \tau + ((\varepsilon_{ik}))\,\tau$$

where

$$\lambda_\tau^2 = \lambda_2^2\cos^2(\alpha-\theta) + \lambda_1^2\sin^2(\alpha-\theta) = \operatorname{ch}\vartheta - \operatorname{sh}\vartheta\,\cos 2(\alpha-\theta) \quad (4.1)$$

Having first determined the projections of the vector τ in the directions of the principal fibers in the deformed state, the unit vector ν' normal to the contour of the body in the deformed state can easily be found:

$$\lambda_\tau\nu' = \lambda_2\cos(\alpha-\theta)\,\mathbf{I}_1' + \lambda_1\sin(\alpha-\theta)\,\mathbf{I}_2'$$

After the orientation of the boundary surface element of the body with respect to the principal stress directions has been determined, the normal and tangential stresses at the boundary surface element can be expressed in terms of the principal stresses or the symmetric stress invariments:

$$p\lambda_\tau^2\sigma_n = \lambda_\tau^2\sigma - \sqrt{1.5}\,\tau_i\,[\sinh\vartheta - \cosh\vartheta\cos 2(\alpha-\theta)]$$
$$p\lambda_\tau^2\tau_n = -\sqrt{1.5}\,\tau_i\sin 2(\alpha-\theta) \tag{4.2}$$

If the functions θ and ϑ_i are here replaced by their expressions in terms of the stress function U, then a formulation of the boundary conditions for the stress function is obtained. In the solution of actual problems it is recommended to group these conditions so that the limitations are applied to the derivatives of the stress functions along the arc of the body contour.

5. Statement of the Problem in Complex Coordinates. As in classical problems of the theory of elasticity of the plane strain state, the solution of many problems in the theory of finite deformations of an incompressible material is conveniently found in terms of complex coordinates. Let

$$z = x_1 + ix_2, \qquad \bar{z} = x_1 + ix_2$$

and the subscripts z and z will denote partial derivatives of a function with respect to the given coordinates. First of all, transform formula (3.11) to complex variables and obtain

$$\sin 2\theta = -i\frac{p}{\tau_i}(U_{zz} - U_{\bar{z}\bar{z}}), \qquad \cos 2\theta = -\frac{p}{\tau_i}(U_{zz} + U_{\bar{z}\bar{z}})$$
$$\tau_i^2 = 4p^2 U_{zz}U_{\bar{z}\bar{z}}, \qquad \frac{\sigma}{\sqrt{1.5}} = f + 2pU_{z\bar{z}}, \qquad \operatorname{tg} 2\theta = i\,\frac{U_{zz} - U_{\bar{z}\bar{z}}}{U_{zz} + U_{\bar{z}\bar{z}}} \tag{5.1}$$

Using these expressions in the displacement compatibility equation (2.7), and changing to complex variables, we find

$$p\left(\frac{\sinh\vartheta}{\tau_i}U_{\bar{z}\bar{z}}\right)_{zz} + p\left(\frac{\sinh\vartheta}{\tau_i}U_{zz}\right)_{\bar{z}\bar{z}} + (\cosh\vartheta)_{z\bar{z}} + i\,[\theta_z(\cosh\vartheta)_{\bar{z}} - \theta_{\bar{z}}(\cosh\vartheta)_z] = 0$$

From (5.1) it can easily be verified that

$$i\theta_z = \frac{p^2}{\tau_i^2}(U_{zz}U_{z\bar{z}\bar{z}} - U_{\bar{z}\bar{z}}U_{zzz}), \qquad i\theta_{\bar{z}} = \frac{p^2}{\tau_i^2}(U_{\bar{z}\bar{z}\bar{z}}U_{zz} - U_{zz\bar{z}}U_{\bar{z}\bar{z}})$$

and thus the equation to be solved is in the form

$$\left(\frac{\sinh\vartheta}{\tau_i}U_{\bar{z}\bar{z}}\right)_{zz} + \left(\frac{\sinh\vartheta}{\tau_i}U_{zz}\right)_{\bar{z}\bar{z}} + \frac{1}{p}(\cosh\vartheta)_{z\bar{z}} +$$
$$+ p\,\frac{\sinh\vartheta}{\tau_i}\,\frac{d\vartheta}{d\tau_i}(U_{zz\bar{z}}U_{z\bar{z}\bar{z}} - U_{zzz}U_{\bar{z}\bar{z}\bar{z}}) = 0 \tag{5.2}$$

If, in particular, the assumptions of the classical theory of elasticity $\sinh \vartheta = \vartheta$, $\cosh \vartheta = 1$, $\sinh \vartheta / \tau_i = G^{-1}$ are used, and components of the order of magnitude of the strains as compared to unity are neglected in equation (5.3), this equation can easily be reduced to the biharmonic equation.

To represent the static boundary conditions in terms of complex variables, let us turn to formulas (4.2) and on the basis of (5.1) state them as follows:

$$p\lambda_\tau^2\sigma_n = \sigma\cosh\vartheta - \sqrt{1.5}\,\tau_i\cosh\vartheta +$$

$$+\frac{p}{\tau_i}\,(\sigma\sinh\vartheta - \sqrt{1.5}\,\tau_i\cosh\vartheta)\,(U_{zz}e^{2i\alpha} + U_{\bar z\bar z}e^{-2i\alpha})$$

$$ip\lambda_\tau^2\tau_n = \sqrt{1.5}\,p\,(U_{zz}e^{2i\alpha} - U_{\bar z\bar z}e^{-2i\alpha})$$

Hence we can easily find

$$\frac{\lambda_\tau^2\sigma_n}{\sqrt{1.5}\cosh\vartheta} = 2U_{z\bar z} - U_{zz}e^{2i\alpha} - U_{\bar z\bar z}e^{-2i\alpha} + \frac{1}{p}f +$$
$$+p\,\frac{\operatorname{th}\vartheta}{\tau_i}\left[4U_{zz}U_{\bar z\bar z} + \left(\frac{1}{p}f + 2U_{z\bar z}\right)(U_{zz}e^{2i\alpha} + U_{\bar z\bar z}e^{-2i\alpha})\right] \qquad (5.3)$$
$$i\,\frac{\lambda_\tau^2\tau_n}{\sqrt{1.5}} = U_{zz}e^{2i\alpha} - U_{\bar z\bar z}e^{-2i\alpha}$$

On the other hand, the derivatives of the stress function along the arc of the contour are given by the formula

$$\frac{dU}{dS} = ie^{-i\alpha}(U_z e^{2i\alpha} - U_{\bar z})$$

Thus, the values of the derivatives of the stress function at the boundary and the values of the normal and tangential boundary stresses are related as follows:

$$2e^{i\alpha}\,\frac{dU_z}{ds} = -\frac{\lambda_\tau^2}{\sqrt{1.5}}\left(\tau_n + i\,\frac{\sigma_n}{\sqrt{1.5}}\right) +$$
$$+ip\left\{\frac{f}{p^2} - \frac{\operatorname{th}\vartheta}{\tau_i}\left[-4U_{zz}U_{\bar z\bar z} + \left(\frac{f}{p} + 2U_{z\bar z}\right)(U_{zz}e^{2i\alpha} + U_{\bar z\bar z}e^{-2i\alpha})\right]\right\} \qquad (5.4)$$

This represents the formulation of the static boundary conditions.

In application it often becomes convenient to apply isometric curvilinear coordinates, which can be obtained as a result of conformal mapping:

$$\zeta = \zeta(z)$$

where

$$\frac{\partial}{\partial z} = \zeta'\,\frac{\partial}{\partial\zeta}, \quad \frac{\partial^2}{\partial z^2} = \zeta'\,\frac{\partial^2}{\partial\zeta^2} + \zeta'\zeta''\frac{\partial}{\partial\zeta}, \quad \frac{\partial}{\partial\bar z} = \bar\zeta'\,\frac{\partial}{\partial\bar\zeta}$$

and the compatibility equation (5.2) becomes

$$(\zeta'\bar{\zeta}')^2\left[\left(\frac{\sinh\vartheta}{\tau_i}U_{\zeta\zeta}\right)_{\bar{\zeta}\bar{\zeta}}+\left(\frac{\sinh\vartheta}{\tau_i}U_{\bar{\zeta}\bar{\zeta}}\right)_{\zeta\zeta}\right]+\bar{\zeta}'^2\bar{\zeta}'\zeta''\left[\left(\frac{\sinh\vartheta}{\tau_i}U_{\zeta}\right)_{\bar{\zeta}\bar{\zeta}}+\right.$$

$$\left.+\left(\frac{\sinh\vartheta}{\tau_i}U_{\bar{\zeta}\bar{\zeta}}\right)_{\zeta}\right]+\zeta'^2\bar{\zeta}'\bar{\zeta}''\left[\left(\frac{\sinh\vartheta}{\tau_i}U_{\bar{\zeta}}\right)_{\zeta\zeta}+\left(\frac{\sinh\vartheta}{\tau_i}U_{\zeta\zeta}\right)_{\bar{\zeta}}\right]+$$

$$+\frac{1}{p}\zeta'\bar{\zeta}'(\cosh\vartheta)_{\zeta\bar{\zeta}}+\zeta'\bar{\zeta}'\zeta''\bar{\zeta}''\left[\left(\frac{\sinh\vartheta}{\tau_i}U_{\zeta}\right)_{\bar{\zeta}}+\left(\frac{\sinh\vartheta}{\tau_i}U_{\bar{\zeta}}\right)_{\zeta}\right]+ \tag{5.5}$$

$$+p\frac{\sinh\vartheta}{\tau_i}\frac{d\vartheta}{d\tau_i}\{(\zeta'\bar{\zeta}')^2(\zeta'U_{\zeta\zeta\bar{\zeta}}+\zeta''U_{\zeta\bar{\zeta}})(\bar{\zeta}'U_{\zeta\bar{\zeta}\bar{\zeta}}+\bar{\zeta}''U_{\zeta\bar{\zeta}})-$$

$$-\bar{\zeta}'\zeta'[\bar{\zeta}'^2U_{\zeta\zeta\zeta}+3\zeta'\zeta''U_{\zeta\zeta}+(\zeta''^2+\zeta'\zeta''')U_{\zeta}][\bar{\zeta}'^2U_{\bar{\zeta}\bar{\zeta}\bar{\zeta}}+$$

$$+3\bar{\zeta}'\bar{\zeta}''U_{\bar{\zeta}\bar{\zeta}}+(\bar{\zeta}''^2+\bar{\zeta}''\bar{\zeta}''')U_{\bar{\zeta}}]\}=0$$

Here the strain intensity is assumed to be represented in terms of the stress intensity and finally in terms of the stress function, since

$$\dot{\tau}_i^2=4p^2(U_{\zeta}\zeta''+U_{\zeta\zeta}\zeta'^2)(U_{\bar{\zeta}}\bar{\zeta}''+U_{\bar{\zeta}\bar{\zeta}}\bar{\zeta}'^2)$$

The boundary conditions are also subject to the same transformation.

Let us study, as an example, the integration of the basic equation using an arbitrary law of change of shape. Since only an axisymmetric deformation is desired, we shall seek a stress function which depends only upon the radial distance of a point in the body:

$$U=\Phi(v),\qquad v=z\bar{z}$$

In the case investigated

$$\tau_i^2=4p^2v^2\Phi''^2$$

$$U_{zz\bar{z}}U_{z\bar{z}\bar{z}}-U_{zzz}U_{\bar{z}\bar{z}\bar{z}}=\frac{1}{2p^2}\frac{d\tau_i^2}{dv} \tag{5.6}$$

Hence the compatibility equation becomes

$$p\left(\frac{\sinh\vartheta}{\tau_i}\Phi''z^2\right)_{zz}+p\left(\frac{\sinh\vartheta}{\tau_i}\Phi''\bar{z}^2\right)_{\bar{z}\bar{z}}+\frac{1}{v}\frac{d}{dv}\left(v^2\frac{d\cosh\vartheta}{dv}\right)=0$$

After changing to the variable v, and after two successive integrations, we obtain

$$2p\frac{\sinh\vartheta}{\tau_i}\Phi''v^2+v\operatorname{ch}\vartheta=C_1v+C_2$$

where G_1 and G_2 are constants of integration.

Using (5.6), we find

$$\sinh^2 \vartheta = \left(\cosh \vartheta - \frac{C_2}{v} - C_1\right)^2$$

and after simple transformations the law of the change of strain intensity becomes

$$\cosh \vartheta = \frac{1}{2}\left[\frac{C_2}{v} + C_1 + \left(\frac{C_2}{v} + C_1\right)^{-1}\right]$$

The constants G_1 and G_2 are to be determined from the boundary conditions.

6. Special Forms of the Basic Equation Corresponding to Different Physical Laws of Change of Shape. Let us first consider two special representations of the physical laws of deformation where the physical properties are characterized by only one constant. If the physical nonlinearity is expressed by the function

$$\tau_i = G \sinh \vartheta, \qquad G = \text{const} \tag{6.1}$$

then it can easily be found that

$$\frac{d\vartheta}{d\tau_i} = \frac{1}{G \cosh \vartheta}, \quad \cosh^2 \vartheta = 1 + 4\nu^2 U_{zz} U_{\bar{z}\bar{z}}, \quad \nu = \frac{p}{G}$$

and the solution equation becomes

$$2\nu U_{zz\bar{z}\bar{z}} + (\cosh \vartheta)_{z\bar{z}} + \frac{\nu^2}{\cosh \vartheta}(U_{zz\bar{z}} U_{z\bar{z}\bar{z}} - U_{zzz} U_{\bar{z}\bar{z}\bar{z}}) = 0$$

Using the expression of the strain intensity in terms of the derivatives of the stress function, we find

$$\begin{aligned}(1 + 4\nu^2 U_{zz} U_{\bar{z}\bar{z}})^{3/2} U_{zz\bar{z}\bar{z}} - 2\nu^3 (U_{zz} U_{\bar{z}\bar{z}})_z (U_{zz} U_{\bar{z}\bar{z}})_{\bar{z}} + \\ + \nu (1 + 4\nu^2 U_{zz} U_{\bar{z}\bar{z}}) \ (U_{zz} U_{\bar{z}\bar{z}})_{z\bar{z}} + \tfrac{1}{2} U_{zz\bar{z}} U_{z\bar{z}\bar{z}} - \tfrac{1}{2} U_{zzz} U_{\bar{z}\bar{z}\bar{z}}] = 0\end{aligned} \tag{6.2}$$

A more widely-occurring physical nonlinearity is of the type

$$\tau_i = G \tan \vartheta, \qquad G = \text{const} \tag{6.3}$$

In this case we find

$$\begin{aligned}\frac{d\vartheta}{d\tau_i} = \frac{\cosh^2 \vartheta}{G}, \qquad \frac{\sinh \vartheta}{\tau_i} = \frac{\cosh \vartheta}{G}, \qquad \mu = \frac{p}{G} \\ \frac{1}{\cosh^2 \vartheta} = 1 - 4\mu^2 U_{zz} U_{\bar{z}\bar{z}}, \qquad f = \tfrac{1}{2} G \ln\left(1 - \frac{\tau_i^2}{G^2}\right)\end{aligned} \tag{6.4}$$

and the compatibility equation

$$\mu\,[(\cosh \vartheta\, U_{zz})_{\bar{z}\bar{z}} + (\cosh \vartheta\, U_{\bar{z}\bar{z}})_{zz}] + (\cosh \vartheta)_{z\bar{z}} + \mu^2 \cosh^3 \vartheta\, (U_{zz\bar{z}} U_{z\bar{z}\bar{z}} - U_{zzz} U_{\bar{z}\bar{z}\bar{z}}) = 0 \tag{6.5}$$

is expressed in the following form:

$$U_{zz\bar{z}\bar{z}} + \sum_{k=1}^{4} \mu^k L_{k+1}(U) = 0 \tag{6.6}$$

Here a special notation for the nonlinear differential operators for the stress function was used:

$$\begin{aligned}
L_2(U) &= t_{z\bar{z}} + \tfrac{1}{2} U_{zz\bar{z}} U_{\bar{z}\bar{z}z} - \tfrac{1}{2} U_{zzz} U_{\bar{z}\bar{z}\bar{z}}, \qquad t = U_{zz} U_{\bar{z}\bar{z}} \\
L_3(U) &= -8tU_{zz\bar{z}\bar{z}} + 2t_z U_{z\bar{z}\bar{z}} + 2t_{\bar{z}} U_{zz\bar{z}} + t_{zz} U_{\bar{z}\bar{z}} + t_{\bar{z}\bar{z}} U_{zz} \\
L_4(U) &= -4tL_2(U) + 6t(U_{zz\bar{z}} U_{z\bar{z}\bar{z}} - U_{zzz} U_{\bar{z}\bar{z}\bar{z}}) + \\
&\quad + 6U_{\bar{z}\bar{z}}^2 U_{zzz} U_{zz\bar{z}} + 6U_{zz}^2 U_{\bar{z}\bar{z}\bar{z}} U_{z\bar{z}\bar{z}} \\
L_5(U) &= 16t^2 U_{zz\bar{z}\bar{z}} - 4t\,[2t_z U_{z\bar{z}\bar{z}} + 2t_{\bar{z}} U_{zz\bar{z}} + \\
&\quad + t_{zz} U_{\bar{z}\bar{z}} + t_{\bar{z}\bar{z}} U] + 6\,[U_{\bar{z}\bar{z}} (t_z)^2 + U_{zz} (t_{\bar{z}})^2]
\end{aligned} \tag{6.7}$$

The structure of the basic equation (6.2) or (6.6) and the boundary conditions obtained by the application of (5.4) indicates that it is possible to express the solutions of specific problems in terms of expansions in series of powers of the parameter ν or μ:

$$U = U^{(0)} + \mu U^{(1)} + \mu^2 (U)^{(2)} + \ldots \tag{6.8}$$

Here the determination of the "zero-th" approximation becomes equivalent to the solution of the problem of the classical theory of elasticity, which is the generation of a biharmonic function satisfying the boundary conditions

$$U_{zz\bar{z}\bar{z}}^{(0)} = 0 \tag{6.9}$$

The following differential equations are obtained for the determination of successive approximations

$$U_{zzz\bar{z}}^{(1)} + L_2(U^{(0)}) = 0 \tag{6.10}$$

$$\begin{aligned}
&U_{zz\bar{z}\bar{z}}^{(2)} + L_2(U^{(0)}) + (U_{zz}^{(0)} U_{\bar{z}\bar{z}}^{(1)} + U_{\bar{z}\bar{z}}^{(0)} U_{zz}^{(1)})_{z\bar{z}} + \\
&+ \tfrac{1}{2} (U_{zz\bar{z}}^{(0)} U_{z\bar{z}\bar{z}}^{(1)} + U_{z\bar{z}\bar{z}}^{(0)} U_{zz\bar{z}}^{(1)}) - \tfrac{1}{2} (U_{zzz}^{(0)} U_{\bar{z}\bar{z}\bar{z}}^{(1)} + U_{\bar{z}\bar{z}\bar{z}}^{(0)} U_{zzz}^{(1)}) = 0
\end{aligned} \tag{6.11}$$

From this it follows that for the determination of each of the successive approximations it is necessary to find the particular solution of a nonhomogeneous fourth-order equation and then to determine the biharmonic function from the boundary values. The Muskhelishvili method is recommended [4] for the solution of problems formulated in this fashion.

In conclusion, it should be noted that the "zero-th" and the "first" approximations for the cases of physical nonlinearity of type (6.1) or (6.3) are the same. Thus the known [5] methods proposed for solving problems of the nonlinear theory of elasticity with a plane deformed state require justification.

7. Example of Analysis of Stress Concentration Near a Circular Cylindrical Cavity in an Inifinite Body. Let us study an infinite body having a circular cylindrical cavity. The axis x_3 coincides with the direction of the axis of the cavity of radius R, and we will thus study the state of the plane x_1, x_2. The state of stress of the body at infinity is assumed to be given. Without loss of generality of this state we assume the body to be compressed by a load at infinity of intensity $2pq\sqrt{1.5}$, uniformly distributed along axis x_2. It is necessary to find the stress distribution around the stress-free surface of the cavity.

From the given stresses at infinity

$$\sigma_1^{\infty} = 0, \qquad \sigma_2^{\infty} = 2\sigma_3^{\infty} = -pq2\sqrt{1.5}$$

in the case of a physical nonlinearity of type (6.3) we compute the symmetric invariants

$$\tau_i^{\infty} = pq, \quad \frac{\sigma}{\sqrt{1.5}} = -pq, \quad f^{\infty} = -\frac{1}{2}p\frac{1}{\mu}\ln(1-\mu^2q^2)$$

From this, the formulation of the conditions for the stress function at infinity can easily be obtained:

$$U_{zz}^{\infty} = U_{\bar{z}\bar{z}}^{\infty} = -\frac{1}{2}q, \quad U_{z\bar{z}}^{\infty} = -\frac{1}{2}q + \frac{1}{4\mu}\ln(1-\mu^2q^2)$$

In accordance with the rules of the method of small parameters, we find the conditions at infinity for the successive approximations:

$$(U_{zz}^{(0)})_{\infty} = (U_{\bar{z}\bar{z}}^{(0)})_{\infty} = (U_{z\bar{z}}^{(0)})_{\infty} = -\frac{1}{2}q \tag{7.1}$$

$$(U_{zz}^{(1)})_{\infty} = (U_{\bar{z}\bar{z}}^{(1)})_{\infty} = 0, \qquad (U_{z\bar{z}}^{(1)})_{\infty} = -\frac{1}{4}q^2 \tag{7.2}$$

$$(U_{zz}^{(2)})_{\infty} = (U_{\bar{z}\bar{z}}^{(2)})_{\infty} = (U_{z\bar{z}}^{(2)})_{\infty} = 0, \ldots \tag{7.3}$$

The conditions at the stress-free contour $z = -Re^{i\alpha}$ is obtained on the basis of (5.4) with $ds = -Rd\alpha$:

$$\frac{d}{d\alpha}U_z^{(0)} = 0 \tag{7.4}$$

$$\frac{1}{R}e^{i\alpha}\frac{d}{d\alpha}U_z^{(1)} = i\,[U_{zz}^{(0)}U_{\bar{z}\bar{z}}^{(0)} - U_{z\bar{z}}^{(0)}(U_{zz}^{(0)}e^{2i\alpha} + U_{\bar{z}\bar{z}}e^{-2i\alpha})] \tag{7.5}$$

$$\frac{1}{R}e^{i\alpha}\frac{d}{d\alpha}U_z^{(2)} = i\,[-(U_{z\bar{z}}^{(1)} + U_{zz}^{(0)}U_{\bar{z}\bar{z}}^{(0)})(U_{zz}^{(0)}e^{2i\alpha} + U_{\bar{z}\bar{z}}^{(0)}e^{-2i\alpha}) +$$
$$+ U_{zz}^{(0)}U_{\bar{z}\bar{z}}^{(1)} + U_{\bar{z}\bar{z}}^{(0)}U_{zz}^{(1)} - U_{z\bar{z}}^{(0)}(U_{zz}^{(1)}e^{2i\alpha} + U_{\bar{z}\bar{z}}^{(1)}e^{-2i\alpha})], \ldots \tag{7.6}$$

The biharmonic function $U^{(0)}$, satisfying conditions (7.1) at infinity and condition (7.4) on the contour of the cavity is known [4]. Using the Goursat formula for the representation of the biharmonic function and the representation of analytic functions in terms of power series, we obtain

$$U^{(0)} = \frac{1}{4} q \left[-R^4 \left(\frac{1}{z_2} + \frac{1}{\bar{z}^2} \right) + 2R^2 \left(\frac{z}{\bar{z}} + \frac{\bar{z}}{z} \right) + 2R^2 \ln z\bar{z} - (z + \bar{z})^2 \right] \qquad (7.7)$$

Computing $L_4(U^0)$, we let

$$\widetilde{U}^{(1)} = -\frac{1}{8} q^2 R^2 \left[\frac{R^6}{(z\bar{z})^3} - \frac{2R^4}{(z\bar{z})^2} + R^4 \left(\frac{1}{z\bar{z}^3} + \frac{1}{z^3\bar{z}} \right) + \frac{R^2}{z\bar{z}} + \frac{z^2}{\bar{z}^2} + \frac{\bar{z}^2}{z^2} \right] \qquad (7.8)$$

and find that equation (6.10) will be satisfied if the function $U^{(1)} - U^{(1)}$ is biharmonic. Again use the Goursat formula and let

$$2U^{(1)} = 2\widetilde{U}^{(1)} + \bar{z}\varphi^{(1)}(z) + z\overline{\varphi^{(1)}(z)} + \varkappa^{(1)}(z) + \overline{\varkappa^{(1)}(z)}$$

where $\phi^{(1)}$ and $\kappa^{(1)}$ are analytic functions.

Satisfying conditions (7.2) at infinity, we find

$$\frac{1}{q^2}\varphi^{(1)}(z) = -\frac{1}{4}z + \varphi_{-1}^{(1)}\frac{1}{z} + \varphi_{-3}^{(1)}\frac{1}{z^3} + \ldots$$

$$\frac{1}{q^2}\varkappa^{(1)}(z) = \widetilde{\varkappa}^{(1)}\ln(z) + \varkappa_{-2}^{(1)}\frac{1}{z^2} + \varkappa_{-4}^{(1)}\frac{1}{z^4} + \ldots$$

When computing the boundary values of the derivatives of $U^{(0)}$, we express the condition on the contour $z = -Re^{i\alpha}$ for determining function $U^{(1)}$ in the form

$$\frac{d}{d\alpha} U_z^{(1)} = -\frac{1}{4} i q^2 R\,(1 + 2\cos 2\alpha)^2 e^{-i\alpha}$$

When this condition is satisfied, we find

$$U^{(1)} = \widetilde{U}^{(1)} + q^2 \left[-\frac{1}{4} z\bar{z} + \frac{3}{8} R^2 \left(\frac{z}{\bar{z}} + \frac{\bar{z}}{z} \right) + \frac{1}{3} R^4 \left(\frac{z}{\bar{z}^3} + \frac{\bar{z}}{z^3} \right) - \right.$$

$$\left. - \frac{1}{2} R^2 \ln z\bar{z} + \frac{1}{12} R^4 \left(\frac{1}{z^2} + \frac{1}{\bar{z}^2} \right) - \frac{7}{40} R^6 \left(\frac{1}{z^4} + \frac{1}{\bar{z}^4} \right) \right] \qquad (7.9)$$

In computing the stress at the point $z = -R$, we obtain

$$U_{zz}^{(0)}\big|_{z=-R} = U_{\bar{z}\bar{z}}^{(0)}\big|_{z=-R} = U_{z\bar{z}}^{(0)}\big|_{z=-R} = -\frac{3}{2}q, \qquad U_{zz}^{(1)} = U_{\bar{z}\bar{z}}^{(1)} = -\frac{3}{4}$$

$$U_{z\bar{z}}^{(1)} = -3q^2, \qquad \tau_i = 3pq\left(1 + \frac{1}{2}\mu q\right), \qquad \frac{\sigma}{\sqrt{1.5}} = -3pq\left(1 + \frac{1}{2}\mu q\right)$$

Letting $q = 1/2\sqrt{1.5}$, we find

$$\sigma_2\,|^{\infty} = -p, \qquad \sigma_2\,|_{r=-R} = -p \cdot 3\,(1 + 0.2\mu) \qquad (7.10)$$

Thus, to allow for only the first approximation in the stress function expression leads to an increase of the stress concentration coefficient as compared with the concentration coefficient computed in the classical theory of elasticity.

This again demonstrates that the first approximation allows for only the geometrical nonlinearity. It is necessary to determine $U^{(2)}$ in order to allow for the effect of the physical nonlinearity. The process of determining the second approximation does not differ basically from that described above for computing the first approximation. Leaving out the complicated expression for $U^{(2)}$, we here introduce only the expression for the concentration coefficient as computed on the basis of the second approximation:

$$m = \frac{\sigma_2^{\max}}{\sigma_2^{\infty}} = 3\,(1 + 0.2\,\mu - \mu^2)$$

From this it follows that, starting with a maximum value of $m_{\max} = 3.03$ for $\mu = 0.1$, we can find a lowering of the stress concentration coefficient together with an increase of the intensity of the external forces.

BIBLIOGRAPHY

1. Novozhilov, V.V., *Osnovy nelineinoi teorii uprugosti* (*Fundamentals of the nonlinear theory of elasticity*). Gostekhizdat, 1948.

2. Tolokonnikov, L.A., Uravneniia nelineinoi teorii uprugosti v peremeshcheniiakh (Equations of the nonlinear theory of elasticity in terms of displacements). *PMM* Vol. 21, No. 6, 1957.

3. Leibenzon, L.S., *Kurs teorii uprugosti* (*A course on the theory of elasticity*). Gostekhizdat, 1947.

4. Muskhelishvili, N.I., *Nekotorye osnovnye zadachi matematicheskoi teorii uprugosti* (*Same basic problems of the mathematical theory of elasticity*). Izd. AN SSSR, 1949.

5. Adkins, J.E., Green, A.E. and Shield, R.T., Finite Plane Strain. *Phil. Trans. Roy. Soc. London* Vol. 246, Ser. A, No. 910, 1953.

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